2.8 Computer Graphics

The mathematics of computer graphics is intimately connected with matrix multiplication.

Recall Theorem 10, page 77:

Theorem 10

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \qquad \text{for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact,

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)].$$

Suppose we want to rotate a point by an angle of θ counterclockwise about the origin. Then

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left[\begin{bmatrix} x' \\ y' \end{bmatrix}\right]$$ as in the following diagram:

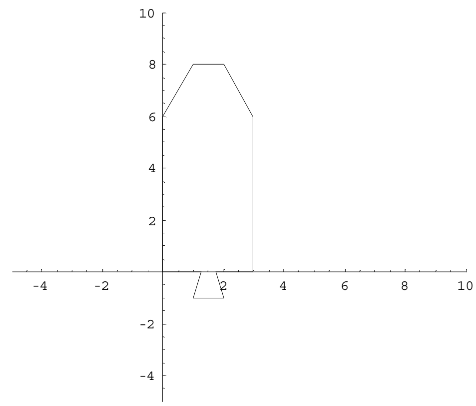
Picture:

Then

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Then $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

EXAMPLE The coordinates of the figure below



are stored in the following matrix:

$$M = \begin{bmatrix} 0 & 1.25 & 1 & 2 & 1.75 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 6 & 8 & 8 & 6 & 0 \end{bmatrix}.$$

Rotations

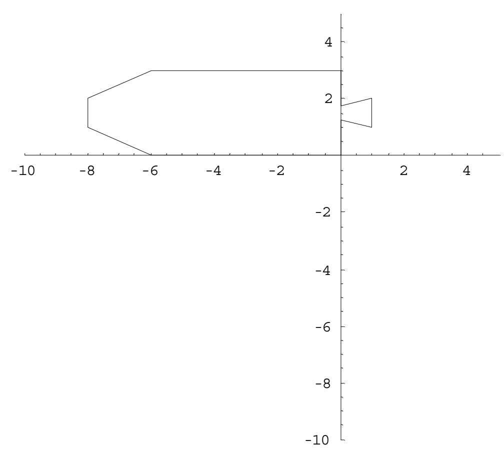
Now suppose we want to rotate the image 90 degrees counterclockwise about the origin. In this case, let $\theta = \frac{\pi}{2}$.

$$R = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then the product RM produces the rotated image:

$$RM = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1.25 & 1 & 2 & 1.75 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 6 & 8 & 8 & 6 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & -6 & -8 & -8 & -6 & 0 \\ 0 & 1.25 & 1 & 2 & 1.75 & 3 & 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$



Scaling

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ where

$$A = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$$

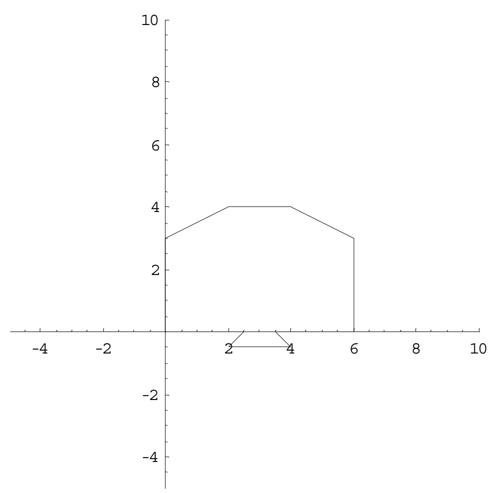
will scale the x by s and y by t.

EXAMPLE To make our figure half as tall and twice as wide, then perform the following:

$$\begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 0 & 1.25 & 1 & 2 & 1.75 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 6 & 8 & 8 & 6 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 2.5 & 2 & 4 & 3.5 & 6 & 6 & 4 & 2 & 0 & 0 \\ 0 & 0 & -.5 & -.5 & 0 & 0 & 3.0 & 4.0 & 4.0 & 3.0 & 0 \end{bmatrix}$$

Corresponding figure:



Homogeneous Coordinates

Translating (i.e. moving) a point from (x, y) to $(x + h, y + k)$ does not directly correspond to a 2×2 linear transformation. The standard way to avoid this difficulty is to use *homogeneous coordinates*.

We associate each point (x, y) in $\mathbf{R}^2$ with the point $(x, y, 1)$ in $\mathbf{R}^3$. We say (x, y) has homogeneous coordinates $(x, y, 1)$. We can then use 3×3 matrices to perform transformations.

EXAMPLE The transformation $(x, y, 1) \mapsto (x+h, y+k, 1)$ is computed via matrix multiplication:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$

EXAMPLE To move our object 10 units right and 10 units up, the following product is formed:

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1.25 & 1 & 2 & 1.75 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 6 & 8 & 8 & 6 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 10 & 11.25 & 11 & 12 & 11.75 & 13 & 13 & 12 & 11 & 10 & 10 \\ 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Corresponding figure:

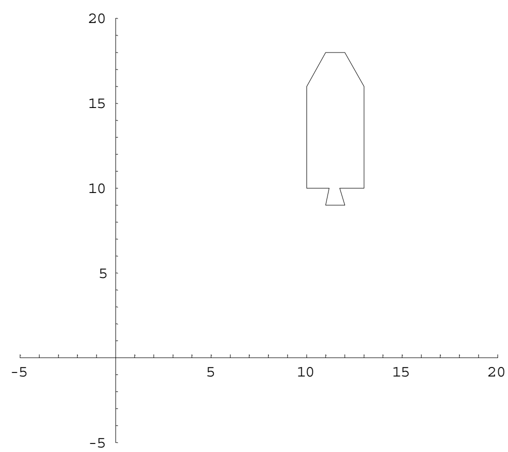


FIGURE 1

Other matrices for linear transformation of homogeneous coordinates:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Counterclockwise rotation about the origin through angle θ.

Scale x by s and y by t

Composite Transformations

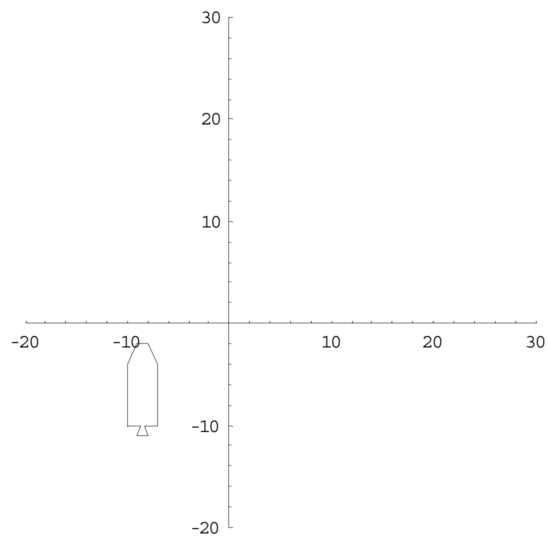
Sometimes moving an object on the screen requires more than one basic transformation. Consider the following example.

EXAMPLE Find a 3×3 matrix that corresponds to clockwise rotation of 90° about the point $(20,20)$.

Consider the matrix corresponding to the image in Figure 1. The first thing we need to do is translate the image left 20 units and down 20 units so that it has the same relative position to $(0,0)$ as it current has to $(20,20)$.

$$\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 11.25 & 11 & 12 & 11.75 & 13 & 13 & 12 & 11 & 10 & 10 \\ 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} -10 & -8.75 & -9 & -8 & -8.25 & -7 & -7 & -8 & -9 & -10 & -10 \\ -10 & -10 & -11 & -11 & -10 & -10 & -4 & -2 & -2 & -4 & -10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

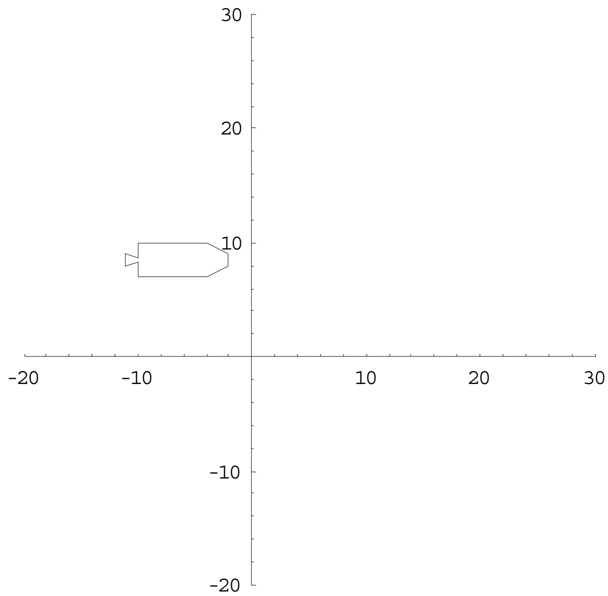
Corresponding figure:



Now let $\theta = -90^\circ$ in $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and will now multiply the last product by this matrix to obtain the matrix corresponding to the rotated image:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 11.25 & 11 & 12 & 11.75 & 13 & 13 & 12 & 11 & 10 & 10 \\ 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} -10 & -10 & -11 & -11 & -10 & -10 & -4 & -2 & -2 & -4 & -10 \\ 10 & 8.75 & 9 & 8 & 8.25 & 7 & 7 & 8 & 9 & 10 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Corresponding image:

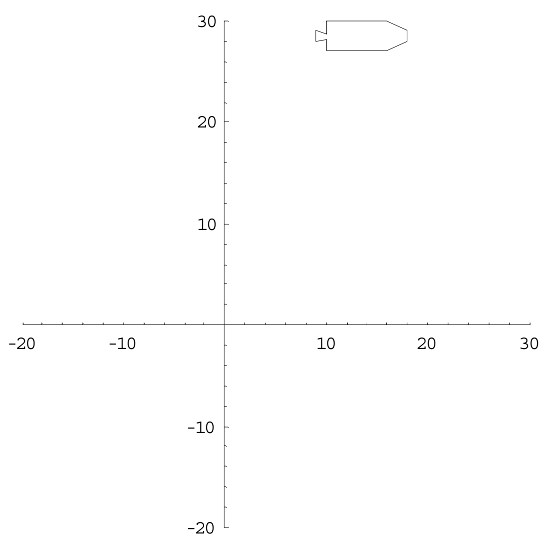


The final step is to translate the image 10 units right and then 10 units up.

$$\begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 11.25 & 11 & 12 & 11.75 & 13 & 13 & 12 & 11 & 10 & 10 \\ 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 30 & 28.75 & 29 & 28 & 28.25 & 27 & 27 & 28 & 29 & 30 & 30 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Correspoinding image:



Multiplying the transformations together will produce the matrix needed to rotate clockwise 90°.

$$\begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 40 \\ 0 & 0 & 1 \end{bmatrix}$$

Creating a Motion Picture

To rotate our figure through an angle of θ about the point $(20, 20)$, we need the following matrix

$$\begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & -20\cos\theta + 20\sin\theta + 20 \\ \sin\theta & \cos\theta & -20\sin\theta - 20\cos\theta + 20 \\ 0 & 0 & 1 \end{bmatrix}$$

Now computing

$$\begin{bmatrix} \cos\theta & -\sin\theta & -20\cos\theta + 20\sin\theta + 20 \\ \sin\theta & \cos\theta & -20\sin\theta - 20\cos\theta + 20 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 11.25 & 11 & 12 & 11.75 & 13 & 13 & 12 & 11 & 10 & 10 \\ 10 & 10 & 9 & 9 & 10 & 10 & 16 & 18 & 18 & 16 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

for $\theta = 0, -\frac{\pi}{12}, -\frac{2\pi}{12}, \ldots, -2\pi,$

we will get 24 frames of a movie showing the figure rotating around the point $(20, 20)$.